Quantum Affine (Super)Algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

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Abstract

We show that the quantum affine algebra $U_q(A_1^{(1)})$ and the quantum affine superalgebra $U_q(C(2)^{(2)})$ admit a unified description. The difference between them consists in the phase factor which is equal to 1 for $U_q(A_1^{(1)})$ and it is equal to -1 for $U_q(C(2)^{(2)})$. We present such a description for the actions of the braid group, for the construction of Cartan-Weyl generators and their commutation relations, as well for the extremal projector and the universal R-matrix. We give also a unified description for the 'new realizations' of these algebras together with explicit calculations of corresponding R-matrices.

1 Introduction

Among variety of all affine Lie (super)algebras[1] (both quantized and non-quantized) the affine (super)algebras of rank 2 play same key role. In the first place, all affine series of the type $A(n|m)^{(1)}$, $B(n|m)^{(1)}$, $C(n)^{(1)}$, $D(n|m)^{(1)}$, $A(2n|2m-1)^{(2)}$, $A(2n-1|2m-1)^{(2)}$,$C(n)^{(2)}$, $D(n|m)^{(2)}$ and $A(2n|2m)^{(4)}$ are started from the affine (super)algebras of rank 2. Secondly, the contragredient Lie (super)algebras of rank 2 are basic structural blocks of any affine (super)algebras of arbitrary rank. This fact permits, for example, to reduce the proof of basic theorems for extremal projector and the universal R-matrix to the proof of such theorems for the (super)algebras of rank 2 (see Refs. [1], [16]–[18], [9]–[13]). Moreover the representation theory of the affine (super)algebras (both quantized and non-quantized) contains some

[1]We conclude the prefix "super" in brackets to stress that the Lie (super)algebras include both the Lie algebras and the own Lie superalgebras.

typical elements of the representation theory of the affine (super)algebras of rank 2. Besides, in applications of the affine (super)algebras, first of all the affine (super)algebras of rank 2 are used by virtue of their simplicity.

In this paper we give detailed description of the quantum untwisted affine algebra $U_q(A_1^{(1)})$ ($\simeq U_q(\hat{sl}(2))$) and the quantum twisted affine superalgebra $U_q(C(2)^{(2)})$ ($\simeq U_q(\widehat{osp}(2|2))^{(2)}$). Moreover our goal is to show that these quantum (super)algebras are described in unified way. Namely, we present in unified way their defining relations and actions of the braid group associated with the Weyl group, the construction of the Cartan-Weyl bases, the complete list of all permutation relations of the Cartan-Weyl generators corresponding to all root vectors and finally the unified formula for their extremal projector and universal R-matrix. We extend also the unified description to so called 'new realizations' of the algebras. Here we present a unified description of the universal R-matrices for corresponding Hopf structures in a multiplicative form as well as in the form of contour integrals. Difference between both considered here quantum (super)algebras is only determined by a phase factor which is equal to 1 for $U_q(A_1^{(1)})$ and it is equal to -1 for $U_q(C(2)^{(2)})$. This situation is similar to the finite-dimensional case. Namely, in the paper [9] it was shown that all quantum (super)algebras $U_q(g)$, where g are the finite-dimensional contragredient Lie (super)algebras of rank 2, are divided into three classes. Each such class is characterized by the same Dynkin diagram and the same reduced root system, provided that we neglect 'colour' of the roots, and all (super)algebras of the same class have the unified defining relations, unified construction and properties of the Cartan-Weyl basis and unified formula for the universal R-matrix. Difference between the (super)algebras of the same class is determined by some phase factors which takes values ± 1 depending on the colour of the nodes of their Dynkin diagram. Concerning the Cartan Weyl bases for the quantum affine algebra $U_q(A_1^{(1)})$ and quantum affine superalgebra $U_q(C(2)^{(2)})$ it should be noted that certain results presented here can be founded in literature separately for the each case (e.g., see Refs. [2], [6], [10]–[13], and [20]).

Basic information about the (super)algebras $A_1^{(1)}$ and $C(2)^{(2)}$ is presented in the tables 1a and 1b (see Refs. [7], [8], [19]). In the table 1a there are listed the standard and symmetric Cartan matrices A and A^{sym}, the corresponding extended symmetric matrices $\bar{A}^{sym}$ and their inverses $(\bar{A}^{sym})^{-1}$, as well as the sets of odd simple roots (odd roots), the Dynkin diagrams (diagram), and the dimensions of these (super)algebras (dim). We remind some elementary definitions of the colour of the roots:

- All even roots are called white roots. A white root is pictured by the white node $\bigcirc$.
- An odd root γ is called a grey root if 2γ is not any root. This root is pictured by the grey node $\otimes$.
- An odd root γ is called a dark root if 2γ is a root. This root is pictured by the dark node $\bullet$.

We also remind the definition of the reduced system of the positive root system Δ_+ for any contragredient (super)algebras of finite growth.

- The system $\underline{\Delta}_+$ is called the reduced system if it is defined by the following way: $\underline{\Delta}_+ = \Delta_+ \backslash \{2\gamma \in \Delta_+ |\, \gamma \text{ is odd}\}$. That is the reduced system $\underline{\Delta}_+$ is obtained from the total system Δ_+ by removing of all doubled roots 2γ where γ is a dark odd root.

The total and reduced root systems of the (super)algebras $A_1^{(1)}$ and $C(2)^{(2)}$ are listed in the table 1b. It is convenient to present the total root systems $\Delta = \Delta_+ \bigcup(-\Delta_+)$ and reduced root systems $\underline{\Delta} = \underline{\Delta}_+ \bigcup(-\underline{\Delta}_+)$ by the pictures: Figs. 1, 2a, 2b. Comparing Fig. 1 and Fig. 2b we see that the reduced root systems of $A_1^{(1)}$ and $C(2)^{(2)}$ coincide if we neglect colour of the roots.

Table 1a

$g(A,\Upsilon)$	$A = A^{sym}$	$\bar{A}^{sym}$	$(\bar{A}^{sym})^{-1}$	odd	diagram
$A_1^{(1)}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$	$\emptyset$	$\delta-\alpha$ α
$C(2)^{(2)}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$	$\{\delta-\alpha, a\}$	$\delta-\alpha$ α

Table 1b

$g(A,\Upsilon)$	Δ_+	$\underline{\Delta}_+$
$A_1^{(1)}$	$\{\alpha,\, n\delta\pm\alpha,\, n\delta \,\vert\, n\in\mathbf{N}\}$	$\{\alpha,\, n\delta\pm\alpha,\, n\delta \,\vert\, n\in\mathbf{N}\}$
$C(2)^{(2)}$	$\{\alpha,\, 2\alpha,\, n\delta\pm\alpha,\, 2n\delta\pm2\alpha,\, n\delta \,\vert\, n\in\mathbf{N}\}$	$\{\alpha,\, n\delta\pm\alpha,\, n\delta \,\vert\, n\in\mathbf{N}\}$

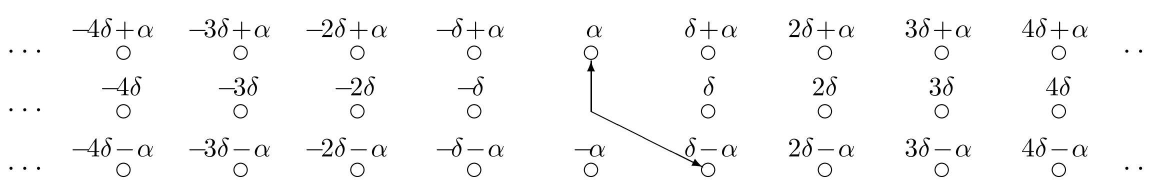

Fig. 1. The total and reduced root system ($\Delta = \underline{\Delta}$) of $A_1^{(1)}(\simeq \widehat{sl}_2)$

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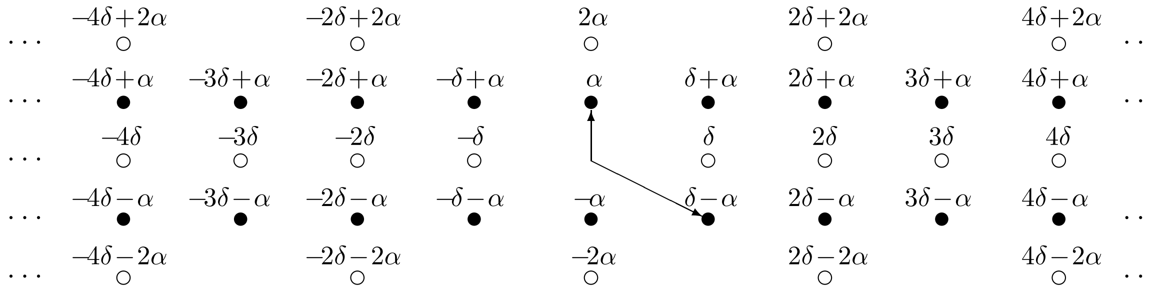

Fig. 2a. The total root system Δ of $C(2)^{(2)}$

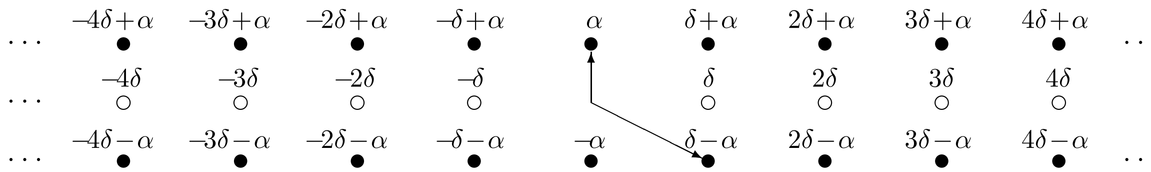

Fig. 2b. The reduced root system $\underline{\Delta}$ of $C(2)^{(2)}$

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2 Defining relations of $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

The quantum (q-deformed) affine (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ are generated by the Chevalley elements $k_{\rm d}^{\pm1} := q^{\pm h_{\rm d}}$, $k_\alpha^{\pm1} := q^{\pm h_\alpha}$, $k_{\delta-\alpha}^{\pm1} := q^{\pm h_{\delta-\alpha}}$, $e_{\pm\alpha}$, $e_{\pm(\delta-a)}$ with the defining relations

$$k_\gamma k_\gamma^{-1} = k_\gamma^{-1} k_\gamma = 1 \ , \qquad [k_\gamma^{\pm1}, k_{\gamma'}^{\pm1}] = 0 \ , \tag{2.1}$$

$$k_\gamma e_{\pm\alpha} k_\gamma^{-1} = q^{\pm(\gamma,\alpha)} e_{\pm\alpha} \ , \qquad k_\gamma e_{\pm(\delta-\alpha)} k_\gamma^{-1} = q^{\pm(\gamma,\delta-\alpha)} e_{\pm(\delta-\alpha)} \ , \tag{2.2}$$

$$[e_\alpha, e_{-\alpha}] = [h_\alpha]_q \ , \qquad [e_{\delta-\alpha}, e_{-\delta+\alpha}] = [h_{\delta-\alpha}]_q \ , \tag{2.3}$$

$$[e_\alpha, e_{-\delta+\alpha}] = 0 \ , \qquad [e_{-\alpha}, e_{\delta-\alpha}] = 0 \ , \tag{2.4}$$

$$[e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm(\delta-\alpha)}]_q]_q]_q = 0 \ , \tag{2.5}$$

$$[[[e_{\pm\alpha}, e_{\pm(\delta-\alpha)}]_q, e_{\pm(\delta-\alpha)}]_q, e_{\pm(\delta-\alpha)}]_q = 0 \ , \tag{2.6}$$

where $(\gamma = {\rm d}, \alpha, \delta-\alpha)$, $({\rm d},\alpha)=0$, $({\rm d},\delta)=1$, and $[h_\beta]_q := (k_\beta - k_\beta^{-1})/(q-q^{-1})$. The brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]_q$ are the super-, and q-super-commutators:

$$\begin{aligned} [e_\beta, e_{\beta'}] &= e_\beta e_{\beta'} - (-1)^{\vartheta(\beta)\vartheta(\beta')} e_{\beta'} e_\beta \ , \\ [e_\beta, e_{\beta'}]_q &= e_\beta e_{\beta'} - (-1)^{\vartheta(\beta)\vartheta(\beta')} q^{(\beta,\beta')} e_{\beta'} e_\beta \ . \end{aligned} \tag{2.7}$$

Here the symbol $\vartheta(\cdot)$ means the parity function: $\vartheta(\beta)=0$ for any even root β, and $\vartheta(\beta)=1$ for any odd root β.

Remark. The left sides of the relations (2.5) and (2.6) are invariant with respect to the replacement of q by q^{-1}. Indeed, if we remove the q-brackets we see that the left sides of (2.5) and (2.6) contain the symmetric functions of q and q^{-1}. This property permits to write the q-commutators in (2.5) and (2.6) in the inverse order, i.e.

$$[[[e_{\pm(\delta-\alpha)},e_{\pm\alpha}]_q,e_{\pm\alpha}]_q,e_{\pm\alpha}]_q \;=\; 0\;, \tag{2.8}$$

$$[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},e_{\pm\alpha}]_q]_q]_q \;=\; 0\;. \tag{2.9}$$

Now we prove the useful proposition.

Proposition 2.1 *(i) In the quantum (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ the following relations*

$$[[e_{\pm\alpha},[e_{\pm\alpha},e_{\pm(\delta-\alpha)}]_q]_q,[[e_{\pm\alpha},[e_{\pm\alpha},e_{\pm(\delta-\alpha)}]_q]_q,[e_{\pm\alpha},e_{\pm(\delta-\alpha)}]_q]_q]_q \;=\; 0\;, \tag{2.10}$$

$$[[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},e_{\pm\alpha}]_q]_q,[[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},e_{\pm\alpha}]_q]_q,[e_{\pm(\delta-\alpha)},e_{\pm\alpha}]_q]_q]_q \;=\; 0\;. \tag{2.11}$$

are fulfilled.
(ii) On the contrary, if the relations (2.1)–(2.4) and (2.10), (2.11) are satisfied then also the relations (2.5), (2.6) are valid.

Thus, the proposition says that under the conditions (2.1)–(2.4) the relations (2.5), (2.6) and (2.10), (2.11) are equivalent.

Proof. Let us assume that the relations (2.1)–(2.6) are fulfilled. We take the relations (2.6) and apply to them the corresponding q-commutator with fourth power of $e_{\pm\alpha}$, i.e. $[e_{\pm\alpha},[e_{\pm\alpha},[e_{\pm\alpha},[e_{\pm\alpha},a_{\pm}]_q\ldots]_q=0$, where $a_{\pm}$ is the left side of (2.6). After tedious calculation we arrive to the relations (2.10). The relations (2.11) are proved in similar way. Namely, the relations $[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},[e_{\pm(\delta-\alpha)},b_{\pm}]_q\ldots]_q=0$, where $b_{\pm}$ is the right-side of (2.5) in the form (2.8), results in (2.11). On the contrary, if the relations (2.1)–(2.4) and (2.10), (2.11) are realized then from the relations $[e_{\mp\alpha},[e_{\mp\alpha},[e_{\mp\alpha},[e_{\mp\alpha},a'_{\mp}]\ldots]=0$ and $[e_{\mp(\delta+\alpha)},[e_{\mp(\delta+\alpha)},[e_{\mp(\delta+\alpha)},[e_{\mp(\delta+\alpha)},b'_{\mp}]\ldots]=0$, where correspondingly $a'_{\mp}$ and $b'_{\mp}$ are the left sides of the relations (2.10) and (2.11), follow the relations (2.5) and (2.6). $\Box$

The standard Hopf structure of the quantum (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ is given by the following formulas for the comultiplication Δ_q and antipode S_q:

$$\begin{aligned} \Delta_q(k_\gamma^{\pm1}) &= k_\gamma^{\pm1}\otimes k_\gamma^{\pm1}\;, & S_q(k_\gamma^{\pm1}) &= k_\gamma^{\mp1}\;, \\ \Delta_q(e_\beta) &= e_\beta\otimes 1+k_\beta^{-1}\otimes e_\beta\;, & S_q(e_\beta) &= -k_\beta e_\beta\;, \\ \Delta_q(e_{-\beta}) &= e_{-\beta}\otimes k_\beta+1\otimes e_{-\beta}\;, & S_q(e_{-\beta}) &= -e_{-\beta}k_\beta^{-1}\;, \end{aligned} \tag{2.12}$$

where $\beta=\alpha,\,\delta-\alpha$; $\gamma=\mathrm{d},\,\beta$.

It is not hard to verify by direct calculations for the defining relations (2.1)–(2.6) that the quantum affine (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$ have the following simple involute (anti)automorphisms.

(i) The non-graded antilinear antiinvolution or conjugation "":*

$$\begin{aligned} (q^{\pm1})^* &= q^{\mp1} , & (k_\gamma^{\pm1})^* &= k_\gamma^{\mp1} , \\ e_\beta^* &= e_{-\beta} , & e_{-\beta}^* &= e_\beta \end{aligned} \tag{2.13}$$

($(xy)^* = y^* x^*$ for $\forall\ x, y \in U_q(g)$).
(ii) The graded antilinear antiinvolution or graded conjugation "‡":

$$\begin{aligned} (q^{\pm1})^\ddagger &= q^{\mp1} , & (k_\gamma^{\pm1})^\ddagger &= k_\gamma^{\mp1} , \\ e_\beta^\ddagger &= (-1)^{\vartheta(\beta)} e_{-\beta} , & e_{-\beta}^\ddagger &= e_\beta \end{aligned} \tag{2.14}$$

($(xy)^\ddagger = (-1)^{\deg x \deg y} y^\ddagger x^\ddagger$ for any homogeneous elements $x, y \in U_q(g)$).
(iii) The Chevalley graded involution ω:

$$\begin{aligned} \omega(q^{\pm1}) &= q^{\mp1} , & \omega(k_\gamma^{\pm1}) &= k_\gamma^{\pm1} , \\ \omega(e_\beta) &= -e_{-\beta} , & \omega(e_{-\beta}) &= -(-1)^{\theta(\beta)} e_\beta . \end{aligned} \tag{2.15}$$

(iv) The Dynkin involution τ which is associated with the automorphism of the Dynkin diagrams of the (super)algebras $A_1^{(1)}$ and $C(2)^{(2)}$:

$$\begin{aligned} \tau(q^{\pm1}) &= q^{\pm1} , & \tau(k_{\mathrm{d}}^{\pm1}) &= k_{\mathrm{d}}^{\pm1} , \\ \tau(k_\beta^{\pm1}) &= k_{\delta-\beta}^{\pm1} , & \tau(k_{-\beta}^{\pm1}) &= k_{-\delta+\beta}^{\pm1} , \\ \tau(e_\beta) &= e_{\delta-\beta} , & \tau(e_{-\beta}) &= e_{-\delta+\beta} . \end{aligned} \tag{2.16}$$

Here in (2.13)–(2.16) $\beta = \alpha, \delta-\alpha$; $\gamma = \mathrm{d}, \beta$.

It should be noted that the graded conjugation "‡" and the Chevalley graded involution ω are involute (anti)automorphism of the fourth order, i.e., for example, $(\omega)^4 = \mathrm{id}$. Note also that the Dynkin involution τ commutes with all other three involutions, i.e. $\tau(x^*) = (\tau(x))^*$, $\tau(x^\ddagger) = (\tau(x))^\ddagger$ and $\omega\tau(x) = \tau\omega(x)$ for any element $x \in U_q(g)$ $(g = A_1^{(1)}, C(2,0)^{(2)})$.

In the next Section we consider a q-analog of automorphisms connected with the Weyl group of the (super)algebras $A_1^{(1)}$ and $C(2,0)^{(2)}$. This q-analog defines actions of the braid group associated with the Weyl group.

3 Braid group actions

We introduce the morphisms T_α and $T_{\delta-\alpha}$ defined by the following formulas:

$$\begin{aligned} T_\alpha(q^{\pm1}) &= q^{\pm1} , & T_\alpha(k_\gamma^{\pm1}) &= k_\gamma^{\pm1} k_\alpha^{\mp\frac{2(\alpha,\gamma)}{(\alpha,\alpha)}} , \\ T_\alpha(e_\alpha) &= -e_{-\alpha} k_\alpha , & T_\alpha(e_{-\alpha}) &= -(-1)^{\theta(\alpha)} k_\alpha^{-1} e_\alpha , \\ T_\alpha(e_{\delta-a}) &= \frac{1}{a}[e_\alpha, [e_\alpha, e_{\delta-a}]_q]_q , \\ T_\alpha(e_{-\delta+a}) &= \frac{(-1)^{\theta(\alpha)}}{a}[[e_{-\delta+\alpha}, e_{-\alpha}]_{q^{-1}}, e_{-\alpha}]_{q^{-1}} , \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
T_{\delta-\alpha}(q^{\pm1}) &= q^{\pm1}\ , & T_{\delta-\alpha}(k_{\gamma}^{\pm1}) &= k_{\gamma}^{\pm1}k_{\delta-\alpha}^{\mp\frac{2(\delta-\alpha,\gamma)}{(\alpha,\alpha)}}\ ,\\
T_{\delta-\alpha}(e_{\delta-\alpha}) &= -e_{-\delta+\alpha}k_{\delta-a}\ , & T_{\delta-\alpha}(e_{-\delta+\alpha}) &= -(-1)^{\theta(\alpha)}k_{\delta-a}^{-1}e_{\delta-\alpha},\\
T_{\delta-\alpha}(e_{\alpha}) &= \frac{1}{a}[e_{\delta-\alpha},[e_{\delta-\alpha},e_{\alpha}]_{q}]_{q}\ , \\
T_{\delta-\alpha}(e_{-\alpha}) &= \frac{(-1)^{\theta(\alpha)}}{a}[e_{-\alpha},e_{-\delta+\alpha}]_{q^{-1}},e_{-\delta+\alpha}]_{q^{-1}},
\end{aligned}
\tag{3.2}
$$

where $\gamma=\mathrm{d},\alpha,\delta-\alpha$. It is not difficult to prove by direct verification that the morphisms T_{α}^{-1} and $T_{\delta-\alpha}^{-1}$ given by

$$
\begin{aligned}
T_{\alpha}^{-1}(q^{\pm1}) &= q^{\pm1}\ , & T_{\alpha}^{-1}(k_{\gamma}^{\pm1}) &= k_{\gamma}^{\pm1}k_{\alpha}^{\mp\frac{2(\alpha,\gamma)}{(\alpha,\alpha)}},\\
T_{\alpha}^{-1}(e_{\alpha}) &= -(-1)^{\theta(\alpha)}k_{\alpha}^{-1}e_{-\alpha}\ , & T_{\alpha}^{-1}(e_{-\alpha}) &= -e_{\alpha}k_{\alpha}\ ,\\
T_{\alpha}^{-1}(e_{\delta-a}) &= \frac{1}{a}[[e_{\delta-\alpha},e_{\alpha}]_{q},e_{\alpha}]_{q}\ ,\\
T_{\alpha}^{-1}(e_{-\delta+a)}) &= \frac{(-1)^{\theta(\alpha)}}{a}[e_{-\alpha},[e_{-\alpha},e_{-\delta+\alpha}]_{q^{-1}}]_{q^{-1}}\ ,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
T_{\delta-\alpha}^{-1}(q^{\pm1}) &= q^{\pm1}\ , & T_{\delta-\alpha}^{-1}(k_{\gamma}^{\pm1}) &= k_{\gamma}^{\pm1}k_{\delta-\alpha}^{\mp\frac{2(\delta-\alpha,\gamma)}{(\alpha,\alpha)}}\ ,\\
T_{\delta-\alpha}^{-1}(e_{\delta-\alpha}) &= -(-1)^{\theta(\alpha)}k_{\delta-a}^{-1}e_{-\delta+\alpha}\ , & T_{\delta-\alpha}^{-1}(e_{-\delta+\alpha}) &= -e_{\delta-\alpha}k_{\delta-a}\ ,\\
T_{\delta-\alpha}^{-1}(e_{\alpha}) &= \frac{1}{a}[[e_{\alpha},e_{\delta-\alpha}]_{q},e_{\delta-\alpha}]_{q}\ ,\\
T_{\delta-\alpha}^{-1}(e_{-\alpha}) &= \frac{(-1)^{\theta(\alpha)}}{a}[e_{-\delta+\alpha},[e_{-\delta+\alpha},e_{-\alpha}]_{q^{-1}}]_{q^{-1}}
\end{aligned}
\tag{3.4}
$$

are inverses to T_{α} and $T_{\delta-\alpha}$, i.e.

$$
T_{\alpha}T_{\alpha}^{-1}=T_{\alpha}^{-1}T_{\alpha}=1\ ,\qquad T_{\delta-\alpha}T_{\delta-\alpha}^{-1}=T_{\delta-\alpha}^{-1}T_{\delta-\alpha}=1\ . \tag{3.5}
$$

Here in (3.1)–(3.4) and in what follows we use the notation:

$$
a:=[(\alpha,\alpha)]_{q}=\frac{q^{(\alpha,\alpha)}-q^{-(\alpha,\alpha)}}{q-q^{-1}}\ . \tag{3.6}
$$

Proposition 3.1 *(i) The morphisms T_{α} and $T_{\delta-\alpha}$ (and also T_{α}^{-1} and $T_{\delta-\alpha}^{-1}$) commute with the graded conjugation "$\ddagger$", i.e.*

$$
(T_{\alpha}(x))^{\ddagger}=T_{\alpha}(x^{\ddagger})\ ,\qquad (T_{\delta-\alpha}(x))^{\ddagger}=T_{\delta-\alpha}(x^{\ddagger}) \tag{3.7}
$$

for any element $x\in U_{q}(g)$.
(ii) The morphisms $T_{\alpha}^{\pm1}$ and $T_{\delta-\alpha}^{\pm1}$ are also compatible with the Chevalley graded involution ω, in sense that:

$$
T_{\alpha}\omega=\omega T_{\alpha}^{-1}\ ,\qquad T_{\delta-\alpha}\omega=\omega T_{\delta-\alpha}^{-1}\ . \tag{3.8}
$$

(iii) The morphisms $T_{\alpha}^{\pm1}$ and $T_{\delta-\alpha}^{\pm1}$ are connected with each other by the Dynkin involution τ, in sense that:

$$
T_{\alpha}\tau=\tau T_{\delta-\alpha}\ ,\qquad T_{\alpha}^{-1}\tau=\tau T_{\delta-\alpha}^{-1}\ . \tag{3.9}
$$

This proposition can be proved by direct verification for the Chevalley basis.

Proposition 3.2 *The morphisms T_α and $T_{\delta-\alpha}$ (and also T_α^{-1} and $T_{\delta-\alpha}^{-1}$) are the automorphisms of the quantum (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$.*

Proof. The proposition is proved by direct verification that the defining relations remain valid under the actions of the given morphisms. To this end we apply Proposition 2.1. Note that under the action of T_α the relations (2.4) and (2.5) are transformed into each other, the relation (2.6) is transformed to (2.10). Analogously, under action of $T_{\delta-\alpha}$ the relations (2.4) and (2.6) are transformed into each other, the relations (2.5) are transformed to (2.11). In addition, it is useful to apply the relations (3.7). □

It is easy to see that all these automorphisms $T_\alpha^{\pm1}$ and $T_{\delta-\alpha}^{\pm1}$ are not Hopf algebra automorphisms of $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$, in sense that, e.g., $T_\alpha\otimes T_\alpha\circ\Delta_q\neq\Delta_q\circ T_\alpha$.

In the case of $U_q(g)$, where g is a finite-dimensional simple Lie algebras, the automorphisms of type $T_\alpha^{\pm1}$ and $T_{\delta-\alpha}^{\pm1}$ are called the Lusztig automorphisms [15].

Introduce the following root vectors:

$$\begin{aligned} e_\delta &:= [e_\alpha, e_{\delta-\alpha}]_q\ , & e_{-\delta} &:= [e_{-\delta+\alpha}, e_{-\alpha}]_{q^{-1}},\\ \tilde e_\delta &:= [e_{\delta-\alpha}, e_\alpha]_q\ , & \tilde e_{-\delta} &:= [e_{-\alpha}, e_{-\delta+\alpha}]_{q^{-1}}. \end{aligned}\tag{3.10}$$

It is not difficult to verify that under the actions of the automorphisms $T_\alpha^{\pm1}$ and $T_{\delta-\alpha}^{\pm1}$ the elements $e_{\pm\delta}$ and $\tilde e_{\pm\delta}$ are transformed as follows:

$$\begin{aligned} T_\alpha(\tilde e_{\pm\delta}) &= (-1)^{\theta(\alpha)}e_{\pm\delta}\ , & T_\alpha^{-1}(e_{\pm\delta}) &= (-1)^{\theta(\alpha)}\tilde e_{\pm\delta}\ ,\\ T_{\delta-\alpha}(e_{\pm\delta}) &= (-1)^{\theta(\alpha)}\tilde e_{\pm\delta}\ , & T_{\delta-\alpha}^{-1}(\tilde e_{\pm\delta}) &= (-1)^{\theta(\alpha)}e_{\pm\delta}\ . \end{aligned}\tag{3.11}$$

Therefore

$$\begin{aligned} T_{2\delta}(e_{\pm\delta}) &= e_{\pm\delta}\ , & T_{2\delta}^{-1}(e_{\pm\delta}) &= e_{\pm\delta}\ ,\\ \tilde T_{2\delta}(\tilde e_{\pm\delta}) &= \tilde e_{\pm\delta}\ , & \tilde T_{2\delta}^{-1}(\tilde e_{\pm\delta}) &= \tilde e_{\pm\delta}\ , \end{aligned}\tag{3.12}$$

where the elements $T_{2\delta}$ and $\tilde T_{2\delta}$ called the translation operators are given by

$$\begin{aligned} T_{2\delta} &= T_\alpha T_{\delta-\alpha}\ , & T_{2\delta}^{-1} &= T_{\delta-\alpha}^{-1}T_a^{-1}\ ,\\ \tilde T_{2\delta} &= T_{\delta-\alpha}T_\alpha\ , & \tilde T_{2\delta}^{-1} &= T_\alpha^{-1}T_{\delta-\alpha}^{-1}\ . \end{aligned}\tag{3.13}$$

Proposition 3.3 *The automorphisms $T_\delta:=T_\alpha\tau$, $T_\delta^{-1}:=\tau T_\alpha^{-1}$ and $\tilde T_\delta:=T_{\delta-\alpha}\tau$, $\tilde T_\delta^{-1}:=\tau T_{\delta-\alpha}^{-1}$ are the square roots of the automorphisms of $T_{2\delta}^{\pm1}$ and $\tilde T_{2\delta}^{\pm1}$ correspondingly, i.e.*

$$\begin{aligned} T_\delta^2 &= T_{2\delta}\ , & T_\delta^{-2} &= \tilde T_{2\delta}^{-1}\ ,\\ \tilde T_\delta^2 &= \tilde T_{2\delta}\ , & \tilde T_\delta^{-2} &= \tilde T_{2\delta}^{-1}\ . \end{aligned}\tag{3.14}$$

Moreover

$$\begin{aligned} T_\delta(e_{\delta-\alpha}) &= -e_{-\alpha}k_\alpha\ , & T_\delta(e_{-\delta+\alpha}) &= -(-1)^{\theta(\alpha)}k_\alpha^{-1}e_\alpha,\\ T_\delta^{-1}(e_\alpha) &= -(-1)^{\theta(\alpha)}k_{\delta-\alpha}^{-1}e_{-\delta+\alpha}\ , & T_\delta^{-1}(e_{-\alpha}) &= -e_{\delta-\alpha}k_{\delta-\alpha}\ , \end{aligned}\tag{3.15}$$

$$\begin{aligned} \tilde{T}_\delta(e_\alpha) &= -e_{-\delta+\alpha}k_{\delta-\alpha}\ , & \tilde{T}_\delta(e_{-\alpha}) &= -(-1)^{\theta(\alpha)}k^{-1}_{\delta-\alpha}e_{\delta-\alpha}, \\ \tilde{T}^{-1}_\delta(e_{\delta-\alpha}) &= -(-1)^{\theta(\alpha)}k^{-1}_\alpha e_{-\alpha}\ , & \tilde{T}^{-1}_\delta(e_{-\delta+\alpha}) &= -e_\alpha k_\alpha\ , \end{aligned} \tag{3.16}$$

and also

$$\begin{aligned} T_\delta(e_{\pm\delta}) &= (-1)^{\theta(\alpha)}e_{\pm\delta}\ , & T^{-1}_\delta(e_{\pm\delta}) &= (-1)^{\theta(\alpha)}e_{\pm\delta}\ , \\ \tilde{T}_\delta(\tilde{e}_{\pm\delta}) &= (-1)^{\theta(\alpha)}\tilde{e}_{\pm\delta}\ , & \tilde{T}^{-1}_\delta(\tilde{e}_{\pm\delta}) &= (-1)^{\theta(\alpha)}\tilde{e}_{\pm\delta}\ . \end{aligned} \tag{3.17}$$

Proof. From (3.9) we have that $T_{\delta-\alpha}=\tau T_\alpha\tau$ and therefore $T_{2d} = T_\alpha T_{\delta-\alpha} = T_\alpha\tau T_\alpha\tau = T^2_\delta$. Analogously $T_\alpha = \tau T_{\delta-\alpha}\tau$ and therefore $\tilde{T}_{2d} = T_{\delta-\alpha}T_\alpha = T_{\delta-\alpha}\tau T_{\delta-\alpha}\tau = \tilde{T}^2_\delta$. The formulas (3.15)–(3.17) are trivial.

In the next section we construct the Cartan-Weyl basis and describe its properties in detail.

4 Cartan-Weyl basis for $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

A general scheme for construction of a Cartan-Weyl basis for quantized Lie algebras and superalgebras was proposed in Ref. [17]. The scheme was applied in detail at first for quantized finite-dimensional Lie (super)algebras [9] and then to quantized non-twisted affine algebras [18].

This procedure is bases on a notion of "normal ordering" for the reduced positive root system. For affine Lie (super)algebras this notation was formulated in [16] (see also [17], [10]–[13]). In our case the reduced positive system has only two normal orderings:

$$\alpha, \delta+\alpha, 2\delta+\alpha, \ldots, \infty\delta+\alpha, \delta, 2\delta, 3\delta, \ldots, \infty\delta, \infty\delta-\alpha, \ldots, 3\delta-\alpha, 2\delta-\alpha, \delta-\alpha, \tag{4.1}$$

$$\delta-\alpha, 2\delta-\alpha, 3\delta-\alpha, \ldots, \infty\delta-\alpha, \delta, 2\delta, 3\delta, \ldots, \infty\delta, \infty\delta+\alpha, \ldots, 2\delta+\alpha, \delta+\alpha, \alpha. \tag{4.2}$$

The first normal ordering (4.1) corresponds to "clockwise" ordering for positive roots in Fig. 1, 2b if we start from root α to root $\delta-\alpha$. The inverse normal ordering (4.2) corresponds to "anticlockwise" ordering for the positive roots when we move from $\delta-\alpha$ to α.

In accordance with the normal ordering (4.1) we set

$$e_\delta := [e_\alpha, e_{\delta-\alpha}]_q\ , \qquad e_{-\delta} := [e_{-\delta+\alpha}, e_{-\alpha}]_{q^{-1}}\ , \tag{4.3}$$

$$e_{n\delta+\alpha} := \frac{1}{a}[e_{(n-1)\delta+\alpha}, e_\delta]\ , \qquad e_{-n\delta-\alpha} := \frac{1}{a}[e_{-\delta}, e_{-(n-1)\delta-\alpha}]\ , \tag{4.4}$$

$$e_{(n+1)\delta-\alpha} := \frac{1}{a}[e_\delta, e_{n\delta-\alpha}]\ , \qquad e_{-(n+1)\delta+\alpha} := \frac{1}{a}[e_{-n\delta+\alpha}, e_{-\delta}]\ , \tag{4.5}$$

$$e'_{n\delta} := [e_\alpha, e_{n\delta-\alpha}]_q\ , \qquad e'_{-n\delta} := [e_{-n\delta+\alpha}, e_{-\alpha}]_{q^{-1}}\ , \tag{4.6}$$

where $n = 1, 2, \ldots$, and a is given by the formula (3.6). Analogously for the inverse normal ordering (4.2) we set

$$\tilde{e}_\delta := [e_{\delta-\alpha}, e_\alpha]_q\ , \qquad \tilde{e}_{-\delta} := [e_{-\alpha}, e_{-\delta+\alpha}]_{q^{-1}}, \tag{4.7}$$

$$\tilde{e}_{n\delta+\alpha} := \frac{1}{a}[\tilde{e}_\delta, \tilde{e}_{(n-1)\delta+\alpha}]\ , \qquad \tilde{e}_{-n\delta-\alpha} := \frac{1}{a}[\tilde{e}_{-(n-1)\delta-\alpha}, \tilde{e}_{-\delta}]\ , \tag{4.8}$$

$$\tilde{e}_{(n+1)\delta-\alpha} := \frac{1}{a}[\tilde{e}_{n\delta-\alpha}, \tilde{e}_{\delta}] \ , \qquad \tilde{e}_{-(n+1)\delta+\alpha} := \frac{1}{a}[\tilde{e}_{-\delta}, \tilde{e}_{-n\delta+\alpha}] \ , \tag{4.9}$$

$$\tilde{e}'_{n\delta} := [e_{\delta-\alpha}, \tilde{e}_{(n-1)\delta+\alpha}]_q \ , \qquad \tilde{e}'_{-n\delta} := [e_{-\delta+\alpha}, \tilde{e}_{-(n-1)\delta-\alpha}]_{q^{-1}}, \tag{4.10}$$

where $n = 1, 2, \ldots$. Thus, we have two systems of the Cartan-Weyl generators: 'direct' and 'inverse'. Each such system together with the Cartan generators $k_{\alpha}^{\pm 1}$, $k_{\delta-\alpha}^{\pm 1}$, $e_{\pm\alpha}$, $e_{\pm(\delta-\alpha)}$ are called the q-analog of the Cartan-Weyl basis (or simply the Cartan-Weyl basis) for the quantum (super)algebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$.

Now we consider some properties of these bases. First of all, the explicit construction of the Cartan-Weyl generators (4.3)–(4.6) (or (4.7)–(4.10)) permits easy to find their properties with respect to the (anti)involutions (2.13)–(2.15). For example, it is evident that

$$e^{*}_{\pm\gamma} = e_{\mp\gamma} \ , \qquad \forall \gamma \in \underline{\Delta}_{+} \ . \tag{4.11}$$

and also

$$\begin{aligned} e^{\ddagger}_{n\delta+\alpha} &= (-1)^{(n+1)\theta(\alpha)} e_{-n\delta-\alpha} \ , & e^{\ddagger}_{-n\delta-\alpha} &= (-1)^{n\theta(\alpha)} e_{n\delta+\alpha} \ , \\ e^{\ddagger}_{n\delta-\alpha} &= (-1)^{n\theta(\alpha)} e_{-n\delta+\alpha} \ , & e^{\ddagger}_{-n\delta+\alpha} &= (-1)^{(n-1)\theta(\alpha)} e_{n\delta-\alpha} \ , \\ e^{\ddagger}_{n\delta} &= (-1)^{n\theta(\alpha)} e_{n\delta} \ , & e^{\ddagger}_{-n\delta} &= (-1)^{n\theta(\alpha)} e_{-n\delta} \ . \end{aligned} \tag{4.12}$$

Further, it is easy to see that the 'direct' and 'inverse' Cartan-Weyl generators (4.3)–(4.6) and (4.7)–(4.10) have very simple connection by the Dynkin involution τ:

$$\begin{aligned} \tau(e_{n\delta+\alpha}) &= \tilde{e}_{(n+1)\delta-\alpha} \ , & \tau(\tilde{e}_{n\delta+\alpha}) &= e_{(n+1)\delta-\alpha} & & (n \in \mathbb{Z}) \ , \\ \tau(e_{n\delta-\alpha}) &= \tilde{e}_{(n-1)\delta+\alpha} \ , & \tau(\tilde{e}_{n\delta-\alpha}) &= e_{(n-1)\delta+\alpha} & & (n \in \mathbb{Z}) \ , \\ \tau(e_{n\delta}) &= \tilde{e}_{n\delta} \ , & \tau(\tilde{e}_{n\delta}) &= e_{n\delta} & & (n \neq 0) \ . \end{aligned} \tag{4.13}$$

The transformation properties with respect to the automorphisms T_{α} and $T_{\delta-\alpha}$ can be not hard obtained with the help of (3.1), (3.2), (3.11), and they have the form

$$\begin{aligned} T_{\alpha}(\tilde{e}_{n\delta+\alpha}) &= (-1)^{(n+1)\theta(\alpha)} e_{n\delta-\alpha} \ , & T_{\alpha}(\tilde{e}_{-n\delta-\alpha}) &= (-1)^{n\theta(\alpha)} e_{-n\delta+\alpha} \ , \\ T_{\alpha}(\tilde{e}_{n\delta-\alpha}) &= (-1)^{(n-1)\theta(\alpha)} e_{n\delta+\alpha} \ , & T_{\alpha}(\tilde{e}_{-n\delta+\alpha}) &= (-1)^{n\theta(\alpha)} e_{-n\delta-\alpha} \ , \\ T_{\alpha}(\tilde{e}_{n\delta}) &= (-1)^{n\theta(\alpha)} e_{n\delta} \ , & T_{\alpha}(\tilde{e}_{-n\delta}) &= (-1)^{n\theta(\alpha)} e_{-n\delta} \ , \end{aligned} \tag{4.14}$$

where $n > 0$, and

$$\begin{aligned} T_{\delta-\alpha}(e_{k\delta+\alpha}) &= (-1)^{k\theta(\alpha)} \tilde{e}_{(k+2)\delta-\alpha} \ , & T_{\delta-\alpha}(e_{-k\delta-\alpha}) &= (-1)^{(k+1)\theta(\alpha)} \tilde{e}_{-(k+2)\delta+\alpha} \ , \\ T_{\delta-\alpha}(e_{l\delta-\alpha}) &= (-1)^{l\theta(\alpha)} \tilde{e}_{(l-2)\delta+\alpha} \ , & T_{\delta-\alpha}(e_{-l\delta+\alpha}) &= (-1)^{(l-1)\theta(\alpha)} \tilde{e}_{(-l+2)\delta-\alpha} \ , \\ T_{\delta-\alpha}(e_{m\delta}) &= (-1)^{m\theta(\alpha)} \tilde{e}_{m\delta} \ , & T_{\delta-\alpha}(e_{-m\delta}) &= (-1)^{m\theta(\alpha)} \tilde{e}_{-m\delta} \ , \end{aligned} \tag{4.15}$$

for $k \geq 0$, $l > 1$, $m > 0$. As corollary of the formulas (4.13)–(4.15) we easy find the actions of the translation operators T_{δ} and $\tilde{T}_{\delta}$:

$$\begin{aligned} T_{\delta}(e_{k\delta+\alpha}) &= (-1)^{k\theta(\alpha)} e_{(k+1)\delta+\alpha} \ , & T_{\delta}(e_{-k\delta-\alpha}) &= (-1)^{(k+1)\theta(\alpha)} e_{-(k+1)\delta-\alpha} \ , \\ T_{\delta}(e_{l\delta-\alpha}) &= (-1)^{l\theta(\alpha)} e_{(l-1)\delta-\alpha} \ , & T_{\delta}(e_{-l\delta+\alpha}) &= (-1)^{(l-1)\theta(\alpha)} e_{(-l+1)\delta+\alpha} \ , \\ T_{\delta}(e_{m\delta}) &= (-1)^{m\theta(\alpha)} e_{m\delta} \ , & T_{\delta}(e_{-m\delta}) &= (-1)^{m\theta(\alpha)} e_{-m\delta} \end{aligned} \tag{4.16}$$

for $k\geq 0$, $l>1$, $m>0$, and

$$
\begin{aligned}
\tilde{T}_{\delta}(\tilde{e}_{n\delta+\alpha}) &= (-1)^{(n-1)\theta(\alpha)}\tilde{e}_{(n-1)\delta+\alpha}\ , & \tilde{T}_{\delta}(\tilde{e}_{-n\delta-\alpha}) &= (-1)^{n\theta(\alpha)}\tilde{e}_{(-n+1)\delta+\alpha}\ , \\
\tilde{T}_{\delta}(\tilde{e}_{n\delta-\alpha}) &= (-1)^{(n+1)\theta(\alpha)}\tilde{e}_{(n+1)\delta-\alpha}\ , & T_{\delta}(\tilde{e}_{-n\delta+\alpha}) &= (-1)^{n\theta(\alpha)}\tilde{e}_{-(n+1)\delta+\alpha}\ , \\
\tilde{T}_{\delta}(\tilde{e}_{n\delta}) &= (-1)^{n\theta(\alpha)}\tilde{e}_{n\delta}\ , & \tilde{T}_{\delta}(\tilde{e}_{-n\delta}) &= (-1)^{n\theta(\alpha)}\tilde{e}_{-n\delta}\ ,
\end{aligned}
\tag{4.17}
$$

where $n>0$. (Also see (3.15) and (3.17)). Using the formulas (4.16)–(4.17) we can easy find the actions for the inverse translation operators T_{δ}^{-1}, $\tilde{T}_{\delta}^{-1}$ and $T_{2\delta}^{-1}$, $\tilde{T}_{2\delta}^{-1}$. These actions are not written here. From the relations (4.16)–(4.17) it is clear that the operators $T_{\delta}^{\pm1}$ and $\tilde{T}_{\delta}^{\pm1}$ can be used for construction of the Cartan-Weyl generators (4.3)–(4.6) starting from the Chevalley basis. In the case of the quantum untwisted affine algebras the similar procedure was applied in the paper [2].

Proposition 4.1 *The root vectors (4.3)–(4.6) satisfy the following permutation relations:*

$$
\begin{aligned}
k_{\rm d}e_{n\delta\pm\alpha}k_{\rm d}^{-1} &= q^{n({\rm d},\delta)}e_{n\delta\pm\alpha}, & k_{\rm d}e'_{n\delta}k_{\rm d}^{-1} &= q^{n({\rm d},\delta)}e'_{n\delta}, \\
k_{\gamma}e_{n\delta\pm\alpha}k_{\gamma} &= q^{\pm(\gamma,\alpha)}e_{n\delta\pm\alpha}, & k_{\gamma}e'_{n\delta}k_{\gamma}^{-1} &= e_{n\delta}
\end{aligned}
\tag{4.18}
$$

for any $n\in\mathbb{Z}$ and any $\gamma\in\underline{\Delta}_{+}$, and also

$$
[e_{n\delta+\alpha},e_{-n\delta-\alpha}] = (-1)^{n\theta(\alpha)}\frac{k_{n\delta+\alpha}-k_{n\delta+\alpha}^{-1}}{q-q^{-1}} \qquad (n\geq 0)\ , \tag{4.19}
$$

$$
[e_{n\delta-\alpha},e_{-n\delta+\alpha}] = (-1)^{(n-1)\theta(\alpha)}\frac{k_{n\delta-\alpha}-k_{n\delta-\alpha}^{-1}}{q-q^{-1}} \qquad (n>0)\ ; \tag{4.20}
$$

$$
[e_{n\delta+\alpha},e_{(n+2m-1)\delta+\alpha}]_{q} = (q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e_{(n+l)\delta+\alpha}e_{(n+2m-1-l)\delta+\alpha}, \tag{4.21}
$$

$$
\begin{aligned}
[e_{n\delta+\alpha},e_{(n+2m)\delta+\alpha}]_{q} &= (q_{\alpha}-1)q_{\alpha}^{-m+1}e_{(n+m)\delta+\alpha}^{2}+ \\
&\quad +(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e_{(n+l)\delta+\alpha}e_{(n+2m-l)\delta+\alpha}
\end{aligned}
\tag{4.22}
$$

for any integers $n\geq 0$, $m>0$;

$$
[e_{(n+2m-1)\delta-\alpha},e_{n\delta-\alpha}]_{q} = -(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e_{(n+2m-1-l)\delta-\alpha}e_{(n+l)\delta-\alpha}, \tag{4.23}
$$

$$
\begin{aligned}
[e_{(n+2m)\delta-\alpha},e_{n\delta-\alpha}]_{q} &= -(q_{\alpha}-1)q_{\alpha}^{-m+1}e_{(n+m)\delta-\alpha}^{2}- \\
&\quad -(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e_{(n+l)\delta-\alpha}e_{(n+2m-l)\delta-\alpha}
\end{aligned}
\tag{4.24}
$$

for any integers n, $m>0$;

$$
\begin{aligned}
[e_{-n\delta+\alpha},e_{(n+2m-1)\delta+\alpha}] &= -(-1)^{(n-1)\theta(\alpha)}(q_{\alpha}^{2}-1)\times \\
&\quad \times\sum_{l=n}^{n+m-1}q_{\alpha}^{-l}k_{n\delta-\alpha}e_{(l-n)\delta+\alpha}e_{(n+2m-1-l)\delta+\alpha}+ \\
&\quad +(q_{\alpha}^{2}-1)\sum_{l=1}^{n-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}k_{\delta}^{l}e_{(-n+l)\delta+\alpha}e_{(n+2m-1-l)\delta+\alpha},
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
[e_{-n\delta+\alpha}, e_{(n+2m)\delta+\alpha}] \;=\; & -(-1)^{(n-1)\theta(\alpha)}(q_{\alpha}^{2}-1)\times \\
& \times\sum_{l=n}^{n+m-1} q_{\alpha}^{-l}k_{n\delta-\alpha}e_{(-n+l)\delta+\alpha}e_{(n+2m-1-l)\delta+\alpha}+ \\
& +(q_{\alpha}^{2}-1)\sum_{l=1}^{n-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}k_{\delta}^{l}e_{(-n+l)\delta+\alpha}e_{(n+2m-1-l)\delta+\alpha}- \\
& -(-1)^{(n-1)\theta(\alpha)}(q_{\alpha}-1)q_{\alpha}^{-m-n+1}k_{n\delta-\alpha}e_{m\delta+\alpha}^{2}
\end{aligned}
\tag{4.26}
$$

for any integers $n,\ m\geq 0$*;*

$$
\begin{aligned}
[e_{(n+2m-1)\delta-\alpha}, e_{-n\delta-\alpha}] \;=\; & (-1)^{(n+1)\theta(\alpha)}(q_{\alpha}^{2}-1)\times \\
& \times\sum_{l=n+1}^{n+m-1} q_{\alpha}^{-l}e_{(n+2m-1-l)\delta+\alpha}e_{(l-n)\delta-\alpha}k_{n\delta+\alpha}^{-1}- \\
& -(q_{\alpha}^{2}-1)\sum_{l=1}^{n-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}e_{(n+2m-1-l)\delta-\alpha}e_{(-n+l)\delta-\alpha}k_{\delta}^{-l},
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
[e_{(n+2m)\delta-\alpha}, e_{-n\delta-\alpha}] \;=\; & (-1)^{(n+1)\theta(\alpha)}(q_{\alpha}^{2}-1)\times \\
& \times\sum_{l=n}^{n+m-1} q_{\alpha}^{-l}e_{(n+2m-l)\delta-\alpha}e_{(l-n)\delta+\alpha}k_{n\delta+\alpha}^{-1}- \\
& -(q_{\alpha}^{2}-1)\sum_{l=1}^{n-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}e_{(n+2m-l)\delta-\alpha}e_{(-n+l)\delta-\alpha}k_{\delta}^{-l}+ \\
& +(-1)^{(n-1)\theta(\alpha)}(q_{\alpha}-1)q_{\alpha}^{-m-n+1}e_{m\delta-\alpha}^{2}k_{n\delta+\alpha}^{-1}
\end{aligned}
\tag{4.28}
$$

for any integers $n\geq 0,\ m>0$*;*

$$
[e_{n\delta+\alpha}, e_{m\delta-\alpha}]_{q} \;=\; e'_{(n+m)\delta} \qquad (n\geq 0,\ m>0)\ , \tag{4.29}
$$

$$
[e_{n\delta+\alpha}, e_{-m\delta-\alpha}] \;=\; -(-1)^{(m+1)\theta(\alpha)}e'_{(n-m)\delta}k_{m\delta+\alpha}^{-1} \qquad (n>m\geq 0)\ , \tag{4.30}
$$

$$
[e_{-m\delta+\alpha}, e_{n\delta-\alpha}] \;=\; -(-1)^{(m-1)\theta(\alpha)}k_{m\delta-\alpha}e'_{(n-m)\delta} \qquad (n>m>0)\ , \tag{4.31}
$$

$$
[e'_{n\delta}, e'_{m\delta}] \;=\; [e'_{-n\delta}, e'_{-m\delta}]=0 \qquad (n>0,\ m>0)\ , \tag{4.32}
$$

$$
[e_{n\delta+\alpha}, e'_{m\delta}] = q_{\alpha}^{-m+1}ae_{(n+m)\delta+\alpha}+(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e_{(n+l)\delta+\alpha}e'_{(m-l)\delta} \tag{4.33}
$$

for any integers $n\geq 0,\ m>0$*;*

$$
[e'_{m\delta}, e_{n\delta-\alpha}] = q_{\alpha}^{-m+1}ae_{(n+m)\delta-\alpha}+(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}q_{\alpha}^{-l}e'_{(m-l)\delta}e_{(n+l)\delta-\alpha} \tag{4.34}
$$

for any integers $n,\ m>0$*;*

$$
\begin{aligned}
[e_{-n\delta+\alpha}, e'_{m\delta}] \;=\; & -(-1)^{(n-1)\theta(\alpha)}q_{\alpha}^{-m+1}ak_{n\delta-\alpha}e_{(m-n)\delta+\alpha}- \\
& -(-1)^{(n-1)\theta(\alpha)}(q_{\alpha}^{2}-1)k_{n\delta-\alpha}\sum_{l=n}^{m-1}q_{\alpha}^{-l}e_{(l-n)\delta+\alpha}e'_{(m-l)\delta}+ \\
& +(q_{\alpha}^{2}-1)\sum_{l=1}^{n-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}k_{\delta}^{l}e_{(-n+l)\delta+\alpha}e'_{(m-l)\delta}
\end{aligned}
\tag{4.35}
$$

for any integers $m\geq n>0$;

$$[e_{-n\delta+\alpha},e'_{m\delta}] = (-1)^{m\theta(\alpha)}q_{\alpha}^{-m+1}ak_{\delta}^{m}e_{(-n+m)\delta+\alpha}+ \\ +(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}k_{\delta}^{l}e_{(-n+l)\delta+\alpha}e'_{(m-l)\delta} \tag{4.36}$$

for any integers $n>m>0$;

$$[e'_{m\delta}e_{-n\delta-\alpha}] = -(-1)^{(n+1)\theta(\alpha)}q_{\alpha}^{-m+1}ae_{(m-n)\delta-\alpha}k_{n\delta+\alpha}^{-1}- \\ -(-1)^{(n+1)\theta(\alpha)}(q_{\alpha}^{2}-1)\sum_{l=n+1}^{m-1}q_{\alpha}^{-l}e'_{(m-l)\delta}e_{(l-n)\delta-\alpha}k_{n\delta+\alpha}^{-1}+ \\ +(q_{\alpha}^{2}-1)\sum_{l=1}^{n}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}e'_{(m-l)\delta}e_{(-n+l)\delta-\alpha}k_{\delta}^{-l} \tag{4.37}$$

for any integers $m>n\geq 0$;

$$[e'_{m\delta}e_{-n\delta-\alpha}] = (-1)^{m\theta(\alpha)}q_{\alpha}^{-m+1}ae_{(-n+m)\delta-\alpha}k_{\delta}^{-m}+ \\ +(q_{\alpha}^{2}-1)\sum_{l=1}^{m-1}(-1)^{l\theta(\alpha)}q_{\alpha}^{-l}e'_{(m-l)\delta}e_{(-n+l)\delta-\alpha}k_{\delta}^{-l} \tag{4.38}$$

for any integers $n\geq m>0$.

Here in the relations (4.21)–(4.38) and in what follows $q_{\alpha}:=(-1)^{\theta(\alpha)}q^{(\alpha,\alpha)}$.
Outline of proof. First of all, the formulas (4.18) are trivial. The relations (4.19) and (4.20) are obtained by application the translation operators T_{δ}^{n} and T_{δ}^{-n} to the relations (2.3). Further, in terms of the generators (4.3)–(4.6) the relation (2.5) means that $[e_{\alpha},e_{\delta+\alpha}]_{q}=0$. Applying to it the operator T_{δ}^{n}, we obtain the relation (4.21) for $m=1$. In the case $m>1$ the formulas (4.21) and (4.22) are proved for arbitrary m by induction. If we apply the operator T_{δ}^{-k} to the relations (4.21) and (4.22) for $n=0$, then in the case $k<m$ we obtain the relations (4.25) and (4.26), in the case $m<k<2m$ we obtain the relations which are obtained from (4.27) and (4.28) by the conjugation "$*$", and finally for $k>2m$ we get the relations which are obtained from (4.23) and (4.24) by the conjugation "$*$". Further, the relation (4.29) for $n=0$ is trivial (see (4.6)). Applying to (4.29) with $n=0$ the operators T_{δ}^{n}, we can obtain for any $n>0$ and $m>0$ the relation (4.29) as well as the relation (4.30). The relation (4.31) can be obtained from (4.29) by repeated application of the operator T_{δ}^{-1}. The relations (4.33) in the case $n=0$ and (4.34) in the case $n=1$ are proved by direct verification with the help of the previous results. Repeated application of the operators $T_{\delta}^{\pm1}$ to these relations results in the general case $n,m>0$. The relation (4.32) is proved by direct verification with the help of the relations (4.33) and (4.34). At last, the relations (4.35)–(4.38) can be obtained from (4.33) and (4.34) by repeated application of the operator T_{δ}^{-1}. □

The imaginary root vectors $e'_{n\delta}$ do not satisfy the relations of the type (4.19) and therefore we introduce the new imaginary roots vectors $e_{\pm n\delta}$ by the following (Schur) relations:

$$e'_{n\delta}=\sum_{p_1+2p_2+\ldots+np_n=n}\frac{\big((-1)^{\theta(\alpha)}(q-q^{-1})\big)^{\sum p_i-1}}{p_1!\cdots p_n!}\;e_{\delta}^{p_1}\cdots e_{n\delta}^{p_n}. \tag{4.39}$$

In terms of generating functions

$$\mathcal{E}'(u) \;:=\; (-1)^{\theta(\alpha)}(q-q^{-1})\sum_{n\geq 1} e'_{n\delta}u^{-n}\ , \tag{4.40}$$

$$\mathcal{E}(u) \;=\; (-1)^{\theta(\alpha)}(q-q^{-1})\sum_{n\geq 1} e_{n\delta}u^{-n} \tag{4.41}$$

the relation (4.39) may be rewritten in the form

$$\mathcal{E}'(u) = -1 + \exp\mathcal{E}(u) \tag{4.42}$$

or

$$\mathcal{E}(u) = \ln(1+\mathcal{E}'(u))\ . \tag{4.43}$$

This provides a formula inverse to (4.39)

$$e_{n\delta} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{\big((-1)^{\theta(\alpha)}(q^{-1}-q)\big)^{\sum p_i -1}(\sum_{i=1}^n p_i-1)!}{p_1!\cdots p_n!}\,(e'_\delta)^{p_1}\cdots(e'_{n\delta})^{p_n}. \tag{4.44}$$

The new root vectors corresponding to negative roots are obtained by the Cartan conjugation $(^*)$:

$$e_{-n\delta} = (e_{n\delta})^*\ . \tag{4.45}$$

Proposition 4.2 *The new root vectors $e_{\pm n\delta}$ satisfy the following commutation relations:*

$$[e_{n\delta+\alpha}, e_{m\delta}] = (-1)^{(m-1)\theta(\alpha)}a(m)e_{(n+m)\delta+\alpha} \qquad (n\geq 0,\ m>0)\ , \tag{4.46}$$

$$[e_{m\delta}, e_{n\delta-\alpha}] = (-1)^{(m-1)\theta(\alpha)}a(m)e_{(n+m)\delta-\alpha} \qquad (n,\ m>0)\ , \tag{4.47}$$

$$[e_{-n\delta+\alpha}, e_{m\delta}] = -(-1)^{(n+m)\theta(\alpha)}a(m)\,k_{n\delta-\alpha}e_{(m-n)\delta+\alpha} \qquad (m\geq n>0)\ , \tag{4.48}$$

$$[e_{-n\delta+\alpha}, e_{m\delta}] = (-1)^{\theta(\alpha)}a(m)\,k_\delta^m e_{(-n+m)\delta+\alpha} \qquad (n>m>0)\ , \tag{4.49}$$

$$[e_{m\delta}, e_{-n\delta-\alpha}] = -(-1)^{(n+m)\theta(\alpha)}a(m)e_{(m-n)\delta-\alpha}\,k_{n\delta+\alpha}^{-1} \qquad (m>n\geq 0)\ , \tag{4.50}$$

$$[e_{m\delta}, e_{-n\delta+\alpha}] = (-1)^{\theta(\alpha)}a(m)e_{(-n+m)\delta-\alpha}\,k_\delta^{-m} \qquad (n\geq m>0)\ , \tag{4.51}$$

$$[e_{n\delta}, e_{-m\delta}] = \delta_{nm}\,a(m)\,\frac{k_\delta^m-k_\delta^{-m}}{q-q^{-1}} \qquad (n,\ m>0)\ , \tag{4.52}$$

where

$$a(m) := \frac{q^{m(\alpha,\alpha)}-q^{-m(\alpha,\alpha)}}{m(q-q^{-1})}\ . \tag{4.53}$$

This can be proved by direct calculation, applying the relations of Proposition (4.1) and the actions of the translation operators $T_\delta^{\pm 1}$.

All the relations of Propositions (4.1), (4.2) together with the ones obtained from them by the conjugation describe complete list of the permutation relations of the Cartan-Weyl bases corresponding to the 'direct' normal ordering (4.1). Applying to these relations the Dynkin involution τ, it is easy to obtain these results for the 'inverse' normal ordering (4.2).

5 Extremal projector for $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

A general formula for the extremal projector for quantized contragredient Lie (super)algebras of finite growth was presented in Refs. [17], [10], [11]. Here we specialize this result to our case $U_q(g)$, where $g = A_1^{(1)},\ C(2)^{(2)}$.

By definition, the extremal projector for $U_q(g)$ is a nonzero element $p := p(U_q(g))$ of the Taylor extension $T_q(g)$ of $U_q(g)$ (see Refs. [17], [10], [11]), satisfying the equations

$$e_\alpha\, p = p\, e_{-\alpha} = 0\ , \qquad e_{\delta-\alpha}\, p = p\, e_{-\delta+\alpha} = 0\ , \qquad p^2 = p\ . \tag{5.1}$$

The explicit expression of the extremal projector p for our case $U_q(g)$ can be presented as follows:

$$p = p_+\, p_0\, p_-\ , \tag{5.2}$$

where the factors p_+, p_0 and p_+ have the following form

$$p_+ = \prod_{n\geq 0}^{\rightarrow} p_{n\delta+\alpha}\ , \qquad p_0 = \prod_{n\geq 1} p_{n\delta}\ , \qquad p_- = \prod_{n\geq 1}^{\leftarrow} p_{n\delta-\alpha}\ . \tag{5.3}$$

The elements p_γ are given by the formula

$$p_{n\delta+\alpha} \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)_{\bar{q}_\alpha}!}\, \varphi_{n,m}^{+} e_{-n\delta-\alpha}^{m} e_{n\delta+\alpha}^{m}\ , \tag{5.4}$$

$$p_{n\delta} \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, \varphi_{n,m}^{0} e_{-n\delta}^{m} e_{n\delta}^{m}\ , \tag{5.5}$$

$$p_{n\delta-\alpha} \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)_{\bar{q}_\alpha}!}\, \varphi_{n,m}^{-} e_{-n\delta+\alpha}^{m} e_{n\delta-\alpha}^{m}\ , \tag{5.6}$$

where the coefficients φ_m^+, φ_m^0 and φ_m^- are determined as follows:

$$\varphi_{n,m}^{+} \;=\; \frac{(-1)^{mn\theta(\alpha)}(q-q^{-1})^m q^{-m(\frac{m-1}{4}+n)(\alpha,\alpha)}}{\prod\limits_{r=1}^{m}\Bigl(k_{n\delta+\alpha} q^{(n+\frac{1}{2}+\frac{r}{2})(\alpha,\alpha)} - (-1)^{(r-1)\theta(\alpha)} k_{n\delta+\alpha}^{-1} q^{-(n+\frac{1}{2}+\frac{r}{2})(\alpha,\alpha)}\Bigr)}\ , \tag{5.7}$$

$$\varphi_{n,m}^{0} \;=\; \frac{n^m (q-q^{-1})^{n+m} q^{-mn(\alpha,\alpha)}}{(q^{n(\alpha,\alpha)} - q^{-n(\alpha,\alpha)})^m (k_\delta^n q^{n(\alpha,\alpha)} - k_\delta^{-n} q^{-n(\alpha,\alpha)})^m}\ , \tag{5.8}$$

$$\varphi_{n,m}^{-} \;=\; \frac{(-1)^{m(n-1)\theta(\alpha)}(q-q^{-1})^m q^{-m(\frac{m-5}{4}+n)(\alpha,\alpha)}}{\prod\limits_{r=1}^{m}\Bigl(k_{n\delta-\alpha} q^{(n-\frac{1}{2}+\frac{r}{2})(\alpha,\alpha)} - (-1)^{(r-1)\theta(\alpha)} k_{n\delta-\alpha}^{-1} q^{-(n-\frac{1}{2}+\frac{r}{2})(\alpha,\alpha)}\Bigr)}\ . \tag{5.9}$$

Here in the relations (5.4)–(5.6) and in what follows we use the notation $\bar{q}_\alpha := (-1)^{\theta(\alpha)} q^{-(\alpha,\alpha)}$, and the symbol $(m)_{\bar{q}_\alpha}$ is defined by the formula (6.4).

Acting by the extremal projector p on any highest weight $U_q(g)$-module M we obtain a space $M^0 = pM$ of highest weight vectors for M if pM has no singularities. An effective example of application of the extremal projector for the case of the quantum algebra $U_q(gl(n, \mathbb{C})$ can be found in Ref. [17].

6 Universal R-matrix for $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$

Any quantum (super)algebra $U_q(g)$ is a non-cocommutative Hopf (super)algebra which has the intertwining operator called the universal R-matrix. By definition [5], the universal R-matrix for the Hopf (super)algebra $U_q(g)$ is an invertible element R of the Tylor extension $T_q(g) \otimes T_q(g)$ of $U_q(g) \otimes U_q(g)$ (see Refs. [11]–[13]), satisfying the equations

$$\tilde{\Delta}_q(a) \ = \ R\Delta_q(a)R^{-1} \qquad\qquad \forall \ a \in U_q(g) \ , \tag{6.1}$$

$$(\Delta_q \otimes \mathrm{id})R \ = \ R^{13}R^{23} \ , \qquad (\mathrm{id} \otimes \Delta_q)R = R^{13}R^{12}, \tag{6.2}$$

where $\tilde{\Delta}_q$ is the opposite comultiplication: $\tilde{\Delta}_q = \sigma\Delta_q$, $\sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$ for all homogeneous elements a, $b \in U_q(g)$. In the relation (6.2) we use the standard notations $R^{12} = \sum a_i \otimes b_i \otimes \mathrm{id}$, $R^{13} = \sum a_i \otimes \mathrm{id} \otimes b_i$, $R^{23} = \sum \mathrm{id} \otimes a_i \otimes b_i$ if R has the form $R = \sum a_i \otimes b_i$. We employ the following standard notation for the q-exponential:

$$\exp_q(x) := 1 + x + \frac{x^2}{(2)_q!} + \ldots + \frac{x^n}{(n)_q!} + \ldots = \sum_{n \geq 0} \frac{x^n}{(n)_q!} \ , \tag{6.3}$$

where

$$(n)_q := \frac{q^n - 1}{q - 1} \ . \tag{6.4}$$

A general formula for the universal R-matrix R for quantized contragredient Lie (super)algebras was presented in Refs. [11]–[13]. Here we specialize this result to our case $U_q(g)$, where $g = A_1^{(1)}, \ C(2)^{(2)}$.

The explicit expression of the universal R-matrix R for our case $U_q(g)$ can be presented as follows:

$$R = R_+ R_0 R_- K \ . \tag{6.5}$$

Here the factors K and $R_\pm$ have the following form

$$K \ = \ q^{\frac{1}{(\alpha,\alpha)} h_\alpha \otimes h_\alpha + h_\delta \otimes h_{\mathrm{d}} + h_{\mathrm{d}} \otimes h_\delta} \ , \tag{6.6}$$

$$R_+ \ = \ \prod_{n \geq 0}^{\rightarrow} \mathcal{R}_{n\delta+\alpha}, \qquad R_- = \prod_{n \geq 1}^{\leftarrow} \mathcal{R}_{n\delta-\alpha}. \tag{6.7}$$

The elements $\mathcal{R}_\gamma$ are given by the formula

$$\mathcal{R}_\gamma = \exp_{\bar{q}_\gamma}\Big(A(\gamma)(q - q^{-1})(e_\gamma \otimes e_{-\gamma})\Big) \ , \tag{6.8}$$

where

$$A(\gamma) \ = \ \begin{cases} (-1)^{n\theta(\alpha)} & \text{if } \gamma = n\delta + \alpha \ , \\ (-1)^{(n-1)\theta(\alpha)} & \text{if } \gamma = n\delta - \alpha \ . \end{cases} \tag{6.9}$$

Finally, the factor R_0 is defined as follows

$$R_0 = \exp\Big((q - q^{-1}) \sum_{n > 0} d(n) e_{n\delta} \otimes e_{-n\delta}\Big) \ , \tag{6.10}$$

where $d(n)$ is the inverse to $a(n)$, i.e.

$$d(n) = \frac{n(q - q^{-1})}{q^{n(\alpha,\alpha)} - q^{-n(\alpha,\alpha)}} \ . \tag{6.11}$$

7 The 'new realization'

Let us denote by d the Cartan element $h_{\rm d}$ and by c the Cartan element h_δ, emphasizing that d defines homogeneous gradation of the algebra and $k_\delta = q^{h_\delta}$ is the central element. It will be convenient in the following to add its square roots $q^{\pm\frac{c}{2}} = k_\delta^{\pm\frac{1}{2}}$. Let us introduce the new notations: $e_n := e_{n\delta+\alpha}$ $(n \geq 0)$, $e_{-n} := -(-1)^{(n-1)\theta(\alpha)} k_{-n\delta+\alpha} e_{-n\delta+\alpha}$ $(n > 0)$, and $f_n := -e_{n\delta-\alpha} k_{n\delta-\alpha}$ $(n > 0)$, $f_{-n} := (-1)^{(n+1)\theta(\alpha)} e_{-n\delta-\alpha}$ $(n \geq 0)$. We also put $a_n := e_{n\delta} q^{\frac{nc}{2}}$ $(n \geq 1)$, and $a_{-n} := (-1)^{n\theta(\alpha)} e_{-n\delta} q^{-\frac{nc}{2}}$ $(n \geq 1)$. Collect the elements e_n, f_n $(n \in \mathbb{Z})$ and $a_{\pm n}$ $(n \geq 1)$ into the generating functions ("fields")

$$\begin{aligned} e(z) &= \sum_{n\in\mathbb{Z}} e_n z^{-n}, & \psi_+(z) &= k_\alpha^{-1} \exp\Big((-1)^{\theta(\alpha)}(q-q^{-1}) \sum_{n=1}^{\infty} a_n z^{-n}\Big), \\ f(z) &= \sum_{n\in\mathbb{Z}} f_n z^{-n}, & \psi_-(z) &= k_\alpha \exp\Big((-1)^{\theta(\alpha)}(q^{-1}-q) \sum_{n=1}^{\infty} a_{-n} z^{n}\Big), \end{aligned} \tag{7.1}$$

such that

$$\deg e(z) = \deg f(z) = \theta(\alpha), \qquad \deg \psi_\pm(z) = 0. \tag{7.2}$$

These fields satisfy the following conjugation conditions with respect to graded conjugation "‡":

$$\begin{aligned} (e(z))^\ddagger &= f(z^{-1}) , & (f(z))^\ddagger &= (-1)^{\theta(\alpha)} e(z^{-1}) , \\ (\psi_+(z))^\ddagger &= \psi_-(z^{-1}) , & (\psi_-(z))^\ddagger &= \psi_+(z^{-1}) , \end{aligned} \tag{7.3}$$

and have the following symmetry with respect to the translation operator T_δ:

$$\begin{aligned} T_\delta(e(z)) &= (-1)^{\theta(\alpha)} z e((-1)^{\theta(\alpha)} z) , & T_\delta(f(z)) &= (-1)^{\theta(\alpha)} z^{-1} f((-1)^{\theta(\alpha)} z) , \\ T_\delta(\psi_+(z)) &= q^{-c} \psi_+((-1)^{\theta(\alpha)} z) , & T_\delta(\psi_-(z)) &= q^{c} \psi_-((-1)^{\theta(\alpha)} z) . \end{aligned} \tag{7.4}$$

Proposition 7.1 *In terms of the fields (7.1) the relations of Section 4 can be rewritten in the following compact form*

$$\begin{aligned} [q^c, \text{everything}] &= 0 , \\ u^d \varphi(v) u^{-d} &= \varphi(uv) \end{aligned} \tag{7.5}$$

where $\varphi(v) = e(v), f(v), \psi_\pm(v)$, and also

$$\psi_\pm(u)\psi_\pm(v) = \psi_\pm(v)\psi_\pm(u) , \tag{7.6}$$

$$(u - \bar{q}_\alpha v) e(u) e(v) = (\bar{q}_\alpha u - v) e(v) e(u) , \tag{7.7}$$

$$(u - q_\alpha v) f(u) f(v) = (q_\alpha u - v) f(v) f(u) , \tag{7.8}$$

$$\psi_\pm(u) e(v) (\psi_\pm(u))^{-1} = (-1)^{\theta(\alpha)} \frac{\bar{q}_\alpha q^{\mp\frac{c}{2}} u - v}{q^{\mp\frac{c}{2}} u - \bar{q}_\alpha v} e(v) , \tag{7.9}$$

$$\psi_\pm(u) f(v) (\psi_\pm(u))^{-1} = (-1)^{\theta(\alpha)} \frac{q_\alpha q^{\pm\frac{c}{2}} u - v}{q^{\pm\frac{c}{2}} u - q_\alpha v} f(v) , \tag{7.10}$$

$$(\psi_+(u))^{-1} \psi_-(v) \psi_+(u) (\psi_-(v))^{-1} = \frac{(q^c u - q_\alpha v)(q^{-c} u - \bar{q}_\alpha v)}{(q^c v - \bar{q}_\alpha u)(q^{-c} v - q_\alpha u)} , \tag{7.11}$$

$$[e(u),f(v)]=\frac{1}{q-q^{-1}}\Big(\delta(\frac{u}{v}q^{-c})\psi_-(vq^{c/2})-\delta(\frac{u}{v}q^{c})\psi_+(uq^{c/2})\Big)\ . \tag{7.12}$$

Here in (7.11) $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$, and the brackets $[\cdot,\cdot]$ in the relation (7.12) mean the supercommutator:

$$[e(u),f(v)]=e(u)f(v)-(-1)^{\theta(\alpha)}f(v)e(u)\ . \tag{7.13}$$

Given description is called the 'new realization', or the current realization of the quantum affine superalgebras $U_q(A_1^{(1)})$ and $U_q(C(2)^{(2)})$. It should be noted that the relations (7.1) and (7.6)–(7.12) differ from the corresponding relations of Refs. [3], [5], [20] by replacement q to q^{-1}. The current realization possesses its own graded comultiplication structure, different from (2.12):

$$\begin{aligned}
\Delta_q^{(D)}(c) &= c\otimes 1+1\otimes c\ ,\\
\Delta_q^{(D)}(d) &= d\otimes 1+1\otimes d\ ,\\
\Delta_q^{(D)}(\psi_\pm(z)) &= \psi_\pm(zq^{\pm\frac{c_2}{2}})\otimes\psi_\pm(zq^{\mp\frac{c_1}{2}})\ ,\\
\Delta_q^{(D)}(e(z)) &= e(z)\otimes 1+\psi_-(zq^{\frac{c_1}{2}})\otimes e(zq^{c_1})\ ,\\
\Delta_q^{(D)}(f(z)) &= f(zq^{c_2})\otimes\psi_+(zq^{\frac{c_2}{2}})+1\otimes f(z)\ ,
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
S_q^{(D)}(c) &= -c\ ,\quad S_q^{(D)}(d)=-d\ ,\\
S_q^{(D)}(\psi_\pm(z)) &= (\psi_\pm(z))^{-1}\ ,\\
S_q^{(D)}(e(z)) &= -(\psi_-(zq^{-\frac{c}{2}}))^{-1}e(zq^{-c})\ ,\\
S_q^{(D)}(f(z) &= -f(zq^{-c})(\psi_+(zq^{-\frac{c}{2}}))^{-1}\ ,
\end{aligned} \tag{7.15}$$

$$\varepsilon(c)=\varepsilon(d)=\varepsilon(e(z))=\varepsilon(f(z))=0\ ,\qquad \varepsilon(\psi^\pm(z))=1\ . \tag{7.16}$$

Here $\Delta_q^{(D)}$, $S_q^{(D)}$, and ε are the comultiplication, antipode and counite correspondingly. The two comultiplications Δ_q and $\Delta_q^{(D)}$ are related by the twist [13]:

$$\Delta_q^{(D)}(x)=F^{-1}\Delta(x)F\ , \tag{7.17}$$

where $F=R_+^{21}$, with R_+ given by (6.7)–(6.9), such that the universal R-matrix for the comultiplication $\Delta_q^{(D)}$ equals to

$$\mathcal{R}^{(D)}=R_0R_-KR_+^{21} \tag{7.18}$$

with the factors from (6.5). In the generators e_n, f_n and a_n it can be rewritten as follows

$$\mathcal{R}^{(D)}=\mathcal{K}\bar{\mathcal{R}}\ , \tag{7.19}$$

where

$$\mathcal{K}\ =\ q^{\frac{h_\alpha\otimes h_\alpha}{(\alpha,\alpha)}}q^{\frac{1}{2}(c\otimes d+d\otimes c)}\exp\Big((q-q^{-1})\sum_{n=1}^{\infty}\overline{d}(n)\,a_n\otimes a_{-n}\Big)q^{\frac{1}{2}(c\otimes d+d\otimes c)}\ , \tag{7.20}$$

$$\bar{\mathcal{R}}\ =\ \prod_{n\in\mathbb{Z}}^{\rightarrow}\exp_{\bar{q}_\alpha}\Big((q^{-1}-q)f_{-n}\otimes e_n\Big)\ , \tag{7.21}$$

and

$$\overline{d}(n)=\frac{n(q-q^{-1})}{q_\alpha^n-q_\alpha^{-n}}\ . \tag{7.22}$$

It is possible to give another presentation of the element $\bar{\mathcal{R}}$ in the completed algebras $\bar{U}(g)$, where g is either $A_1^{(1)}$ or $C(2)^{(2)}$ [3], [4]. The completion is done with respect to open neighborhoods of zero $\bar{U}_r=\sum_{s>r}U_s$, where U_s consists of all the elements from $U(g)$ of degree s. The completed algebra acts on (infinite-dimensional) representations of highest weight and admits the series over monomials $x_{i_1}x_{i_2}\cdots x_{i_n}$, $i_1\le i_2...\le i_n$, with $x=e,f,a$ and fixed $\sum i_k$. The matrix coefficients of the products of the currents $e(z_1)e(z_2)\cdots e(z_n)$ and $f(z_1)f(z_2)\cdots f(z_n)$, defined originally as formal series, converge to meromorphic in $\mathbb{C}^n$ functions with the poles at $z_i=0$ and $z_i=q_\alpha^{\mp1}z_j$, $i\le j$.

Let $t(z)=(q-q^{-1})f(z)\otimes e(z)$. As before, we understand the product $t(z_1)\cdots t(z_n)$ as operator-valued meromorphic function in $(\mathbb{C}^*)^n$ with simple poles at $z_i=q_\alpha^{\mp1}z_j$, $i\neq j$. Define

$$\bar{\mathcal{R}}'=1+\sum_{n>0}\frac{1}{n!(2\pi i)^n}\oint\cdots\oint_{D_n}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\,t(z_1)\cdots t(z_n)\ , \tag{7.23}$$

and integration region D_n is defined as $D_n=\Big\{\big|z_i\big|=1,\ i=1,...,n\Big\}$ for $|q|<1$ and, more generally, by

$$D_n=\Big\{\Big|z_i\prod_{\substack{j=1,\dots,n,\\ j\neq i}}(z_i-q_\alpha z_j)\Big|=1,\ i=1,...,n\Big\} \tag{7.24}$$

for any q, such that $q_\alpha^N\neq1$, $N\in\mathbb{Z}\backslash\{0\}$.

Proposition 7.2 *The action of the tensor $\mathcal{R}'=\mathcal{K}\bar{\mathcal{R}}'$ in tensor product of highest weight modules is well defined and coincides with the action of the universal R-matrix (7.19)*

The integrals in (7.20) can be computed explicitly. Let us put by induction

$$t^{(n)}(z)=-\operatorname*{Res}_{z_1=z\bar{q}_\alpha^{-2n-2}}t(z_1)t^{(n-1)}(z)\frac{dz_1}{z_1}, \tag{7.25}$$

where $t^{(1)}(z)=t(z)$. In the components the fields $t^{(n)}(z)$ look as follows:

$$t^{(n)}(z)=C_ne^{(n)}(z)\otimes f^{(n)}(z)\ , \tag{7.26}$$

where

$$\begin{aligned}C_n&=(-1)^{(n-1)\theta(\alpha)}(q-q^{-1})^n\tilde{q}_\alpha^{-\frac{n(n-1)}{2}}(\tilde{q}_\alpha-1)^{n-1}(n-1)_{\tilde{q}_\alpha}!\,(n)_{\tilde{q}_\alpha}!\ ,\\ e^{(n)}(z)&=e(z)e(\bar{q}_\alpha z)e(\bar{q}_\alpha^{\,2}z)\cdots e(\bar{q}_\alpha^{\,n-2}z)e(\bar{q}_\alpha^{\,n-1}z)\ ,\\ f^{(n)}(z)&=f(\bar{q}_\alpha^{\,n-1}z)f(\bar{q}_\alpha^{\,n-2}z)\cdots f(\bar{q}_\alpha^{\,2}z)f(\bar{q}_\alpha z)f(z)\ ,\end{aligned} \tag{7.27}$$

such that

$$\begin{aligned}e^{(n)}(z)&=\sum_{m\in\mathbb{Z}}(z\bar{q}_\alpha^n)^m\sum_{\substack{\lambda_1\ge\dots\ge\lambda_n,\\ \lambda_1+\dots+\lambda_n=m}}\frac{q_\alpha^{\lambda_1+2\lambda_2+\dots n\lambda_n}}{\prod_{j\in\mathbb{Z}}(\lambda'_j-\lambda'_{j+1})_{\bar{q}_\alpha}!}e_{\lambda_n}e_{\lambda_{n-1}}\cdots e_{\lambda_1}\ ,\\ f^{(n)}(z)&=\sum_{m\in\mathbb{Z}}(zq_\alpha)^m\sum_{\substack{\lambda_1\ge\dots\ge\lambda_n,\\ \lambda_1+\dots+\lambda_n=m}}\frac{q_\alpha^{\lambda_1+2\lambda_2+\dots n\lambda_n}}{\prod_{j\in\mathbb{Z}}(\lambda'_j-\lambda'_{j+1})_{q_\alpha}!}f_{\lambda_n}f_{\lambda_{n-1}}\cdots f_{\lambda_1}\ .\end{aligned} \tag{7.28}$$

Here $\lambda'_j = \#k$, such that $\lambda_k \geq j$, and $\tilde{q}_\alpha := q^{(\alpha,\alpha)}$. The product in denominator is finite, since there are only finitely many distinct λ'_j for a given choice of λ_k. Then, repeating the calculations in [4], we get vertex type presentation of the element $\bar{\mathcal{R}}'$:

$$\bar{\mathcal{R}}' = \exp\Bigl(\sum_{n>0} \frac{1}{n} I_n\Bigr) \ , \tag{7.29}$$

where the sequence of operators

$$I_n = \oint \frac{t^{(n)}(z)dz}{2\pi i z} \tag{7.30}$$

commute between themselves:

$$[I_n, I_m] = 0, \qquad n, m > 0.$$

The vertex operator presentation (7.29) is convenient for applications to integrable representations: it is expressed through integrals over the fields, which number is precisely k for level k integrable representations.

8 Final Remarks

The aim of this paper is to describe in unified way with detail the q-deformed untwisted affine algebra $U_q(\widehat{sl}(2)) = U_q(A_1^{(1)})$ and twisted superalgebra $U_q(osp(2|2))^{(2)} = U_q(C(2)^{(2)})$. In order to describe the complete list of quantum affine (super)algebras of rank 2 one should consider some more three quantum affine (super)algebras: $U_q(sl(1|3))^{(4)} = U_q(A(0,2))^{(4)}$, $U_q(sl(3))^{(2)} = U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1|2)) = U_q(B(0,1)^{(1)}$. The Dynkin diagram of the superalgebra $A(0,2)^{(4)}$ has as geometric structure as the (super)algebras $A_1^{(1)}$ and $C(2)^{(2)}$ but in this case the root α is even and $\delta - \alpha$ is odd one, and the sector of imaginary roots has odd roots. Therefore in the case of the quantum superalgebra $A(0,2)^{(4)}$ the relations of the type (4.29)–(4.32) are more complicated and they demand special consideration. The second family of two quantum affine (super)algebras $U_q(A_2^{(2)})$ and $U_q(B(0,1)^{(1)})$ are described by the same Dynkin diagram with different colors of roots. Preliminary results in this direction are given in [14], where in particular the Cartan-Weyl basis of basic affine superalgebra $U_q(\widehat{osp}(1|2))$ is considered. The unified description of three mentioned above quantum affine (super)algebras, analogous to the one given in the present paper, is in preparation.

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